Maximizing Multicast Lifetime with Transmitter-Receiver Power Tradeoff is NP-Hard

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***Abstract*—The problem of maximizing multicast lifetime (MML) in wireless ad hoc networks is reexamined under a recently proposed transmitter-receiver power tradeoff (TRPT) model, for which the energy consumed by a node to reliably receive a bit is inversely proportional to the energy level at which the bit is transmitted. Under the TRPT model, MML was conjectured to be NP-hard. We herein prove the conjecture under the assumption of bounded and discrete power levels.**

***Index Terms*—Wireless ad hoc networks, transmitter-receiver power tradeoff, multicast lifetime, NP-hard**

I. Introduction

An innovative transmit-receive power tradeoff model (TRPT) has been recently introduced, in which when a bit is transmitted at a energy level higher than the minimum required for a reliable reception, the decoder, e.g., turbo decoder, can decode the bit faster and expend less energy [1]. Here, we study the problem of *maximizing multicast lifetime* (MML) [2]–[5] in Wireless Ad Hoc Networks (WANET) under the TRPT model for a single-source multicast problem using a static multicast tree. In [6], a weaker problem (i.e., a multicast tree was given) was solved under the TRPT model, and MML was conjectured to be NP-hard. We herein prove the conjecture under the assumption of bounded and discrete power levels. By contrast, MML with fixed reception energy is in P [6].

II. Energy Model and Problem Statement

Energy consumption per bit is the sum of the reception and transmission energy. A bit transmitted by node u at the energy level j is received (reliably) by node v at the energy level $i \geqslant g_{uv}(j)$, where $g_{uv}(j)$ is a non-increasing function [1]. Specifically, $g_{uv}:\{1,\dots,\Gamma(u)\}\rightarrow\{1,\dots,\Delta(v)+1\}$, where $\Gamma:N\rightarrow\mathbb{N}$ and $\Delta:N\rightarrow\mathbb{N}$ give the number of levels at which a node can transmit and receive, respectively, with N being a set of nodes. The reception level $\Delta(v)+1$ means the node cannot receive the bit. Assuming u has residual battery energy B_u, its lifetime is $\ell_u(i,j)=B_u/(r_u(i)+t_u(j))$, where the non-decreasing functions $r_u(i)$ and $t_u(j)$ give the energy consumed to receive and transmit a bit at level i and j. WANET W is modeled by a tuple $(N,E,B,\Delta,\Gamma,r,t,g)$, where E is the set of directed links; directed link $(u,v)\in E$ *iff* v can reliably receive bits transmitted by u at its highest transmission level.

We assume a single-source multicast problem $\mathtt{mult}(s,D)$, for which bits generated by a *source* node $s\in N$ are delivered to a set $D\subseteq N$ of *destination* nodes. A subgraph of W is a *multicast tree*, denoted by $T=(N'\subseteq N, E'\subseteq E, \psi)$, if it is a tree rooted at s and spanning every node in D; the *energy assignment*, $\psi(T)$, is a vector whose element $\psi_u=\langle\psi_{u,\mathrm{rx}},\psi_{u,\mathrm{tx}}\rangle$, $u\in N'$, is a tuple of reception level $\psi_{u,\mathrm{rx}}$ and transmission level $\psi_{u,\mathrm{tx}}$ such that $\psi_{v,\mathrm{rx}}\geqslant g_{uv}(\psi_{u,\mathrm{tx}})$ for each link $(u,v)\in E'$.

Definition 1: **Lifetime of multicast tree** $T=(N',E',\psi)$ is the duration until the first node dies of battery exhaustion, i.e.,

$$\mathcal{L}(T) \equiv \min_{u\in N'} \ell_u(\psi_{u,\mathrm{rx}},\psi_{u,\mathrm{tx}}) = \min_{u\in N'} B_u/(r_u(\psi_{u,\mathrm{rx}})+t_u(\psi_{u,\mathrm{tx}})).$$

Our optimization problem is as follows:

Maximum multicast lifetime (MML): For $\mathtt{mult}(s,D)$ in W, find a multicast tree T^* s. t. $\mathcal{L}(T^*)=\max_{T\in\mathcal{T}}\mathcal{L}(T)$, where $\mathcal{T}$ is a set of multicast trees w.r.t. $\mathtt{mult}(s,D)$.

III. NP-hardness of MML

Proof outline: First, an auxiliary graph-based optimization problem, *MAL*, is formulated, in which the discrete levels of transmission and reception of every node in W are enumerated; next, we show that MAL is equivalent to MML; then, a decision problem *AIK* equivalent to the decision version of MAL (*MAL$_D$*) is formulated; finally, AIK is proved to be NP-complete by reducing *exact cover by 3-sets* (*X3C*) [7] to AIK:

$$\mathrm{MML} \Leftrightarrow \mathrm{MAL} \underset{\text{optimization}}{\overset{\text{decision}}{\rightleftharpoons}} \mathrm{MAL_D} \Leftrightarrow \mathrm{AIK} \geqslant_p \mathrm{X3C}.$$

A. Energy level representation (auxiliary digraph)

Given W and $\mathtt{mult}(s,D)$, s. t. $s\in N$ and $D\subseteq N$, we construct the auxiliary digraph $G=(V,A,w)$, where V is a set of vertices, A is a set of directed arcs and $w:A\rightarrow\mathbb{R}^+$ is a weight function[1]:

- **V:** consists of sets V_u^{rx} of *reception vertices* and V_u^{tx} of *transmission vertices* for $u\in N$. Vertices $u_i^{\mathrm{rx}}\in V_u^{\mathrm{rx}}$, $1\leqslant i\leqslant\Delta(u)$, and $u_j^{\mathrm{tx}}\in V_u^{\mathrm{tx}}$, $1\leqslant j\leqslant\Gamma(u)$, represent the i^{th} reception and j^{th} transmission level, respectively. V_s^{rx} has, corresponding to zero reception energy, a single vertex s_0^{rx}, called the *origin*, which acts as the root of the arborescence; Additionally, each V_u^{tx}, $u\in D$, includes a special vertex u_0^{tx} corresponding to zero transmission energy called a *sink*.
- **A:** is composed of *intra-node* and *inter-node* arcs w.r.t. W: **intra-node arcs:** for $u\in N$, a set A_u of directed arcs joins each pair of vertices $u_i^{\mathrm{rx}}\in V_u^{\mathrm{rx}}$ and $u_j^{\mathrm{tx}}\in V_u^{\mathrm{tx}}$, implying that u receives at level i and transmits at level j; the weight $w(u_i^{\mathrm{rx}},u_j^{\mathrm{tx}})\equiv 1/\ell_u(i,j)$. **inter-node arcs:** for $(u,v)\in E$, a set $A_{(u,v)}$ of directed arcs connects each pair of vertices $u_j^{\mathrm{tx}}\in V_u^{\mathrm{tx}}$ and $v_i^{\mathrm{rx}}\in V_v^{\mathrm{rx}}$ if $i\geqslant g_{uv}(j)$; the weight $w(u_j^{\mathrm{tx}},v_i^{\mathrm{rx}})\equiv 0$.

The construction of $G=(V,A,w)$ can be done in polynomial time, where $V=\bigcup_{u\in N} V_u^{\mathrm{rx}}\cup V_u^{\mathrm{tx}}$ and $A=\bigcup_{u\in N} A_u \cup \bigcup_{(u,v)\in E} A_{(u,v)}$. The sets V_u^{rx}, V_u^{tx} and A_u constitute the *bipartite digraph* G_u for $u\in N$. E.g., auxiliary digraph in Fig.1(c) is for WANET W in Fig.1(a): each gray disk (bipartite digraph) is for a node in W.

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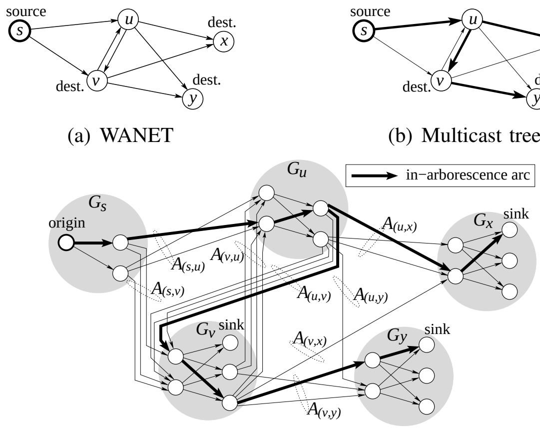
[1] In G, *vertex*, *arc* and *arborescence* mean node, link and tree, respectively.



(a) WANET (b) Multicast tree

(c) Auxiliary digraph with a m-arbor in it

Fig. 1. Each node has two transmission and two reception levels. A m-arbor is depicted in (c) and the corresponding multicast tree is drawn in (b).

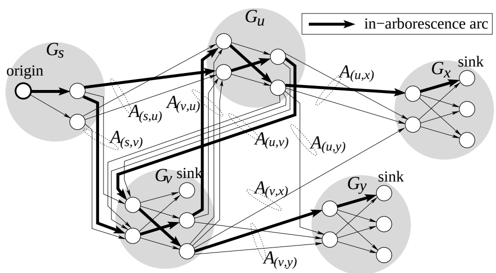


Fig. 2. A non-m-arbor in the auxiliary digraph due to two intra-node arcs in each G_u and G_v.

B. MAL: equivalent of MML in the auxiliary digraph

A multicast tree can be represented as an arborescence in the auxiliary digraph according to the energy assignment, i.e., the reception and transmission levels adopted by each node. However, an arborescence may not necessarily represent a multicast tree. We define here a special type of arborescence and later show that it can be reduced to a multicast tree.

Definition 2: In an auxiliary digraph, a **flat** subgraph contains at most one intra-node arc of each bipartite digraph.

Definition 3: A flat arborescence $H=(V'\subseteq V, A'\subseteq A)$ in $G=(V,A,w)$ is a **multicast arborescence (m-arbor)** if H is rooted at the origin and spans each bipartite digraph that has a sink.

Definition 4: **Lifetime of m-arbor** $H=(V',A')$ is the inverse of the highest arc weight in A', i.e.,

$$\mathcal{L}(H) \equiv \Big(\max_{(u,v)\in A'} w(u,v)\Big)^{-1}.$$

Maximum m-arbor lifetime (MAL): For $\texttt{mult}(s,D)$ in W, seek a m-arbor H^* such that $H^*=\text{argmax}_{H\in\mathcal{H}}\mathcal{L}(H)$, where $\mathcal{H}$ is the set of m-arbors in the auxiliary digraph of W.

For example, Fig. 1(c) depicts a m-arbor, whose corresponding multicast tree is shown in Fig. 1(b). The intra-node arcs included in the m-arbor determine the energy assignment of the multicast tree. Note, a m-arbor does not have to span a sink explicitly; including any intra-node arc of a bipartite digraph implies that the sink of the bipartite digraph is covered by the m-arbor. Also note that an arborescence is not necessarily a m-arbor (see Fig. 2 for a counter-example). While a non-m-arbor does not correspond to any multicast tree, any m-arbor can be transformed to a multicast tree (see lemma 1).

Lemma 1: Let G be the auxiliary digraph of W. For any multicast tree T in W, we can identify a m-arbor H in G such that $\mathcal{L}(T)=\mathcal{L}(H)$, and vice versa.

Proof: First, suppose $T=(N',E',\psi)$ is a multicast tree in W. T can be represented as a m-arbor $H=(V',A')$, where $V'=\{u^{\text{rx}}_{\psi_{u,\text{rx}}}, u^{\text{tx}}_{\psi_{u,\text{tx}}}|u\in N'\}$ and $A'=\{(u^{\text{tx}}_{\psi_{u,\text{tx}}}, v^{\text{rx}}_{\psi_{v,\text{rx}}})|(u,v)\in E'\}\cup\{(u^{\text{rx}}_{\psi_{u,\text{rx}}}, u^{\text{tx}}_{\psi_{u,\text{tx}}})|u\in N'\}$. Conversely, let $H=(V',A')$ be a m-arbor in G. A multicast tree $T=(N',E',\psi)$ can be derived by $N'=\{u|(u^{\text{rx}}_i,u^{\text{tx}}_j)\in A'\}$, $E'=\{(u,v)|(u^{\text{tx}}_j,v^{\text{rx}}_i)\in A'\}$, and $\psi_{u,\text{rx}}=i$ and $\psi_{u,\text{tx}}=j$ for $(u^{\text{rx}}_i,u^{\text{tx}}_j)\in A'$. It can be verified that $\mathcal{L}(H)=\mathcal{L}(T)$ holds in both transformations. ■

Corollary 1: MAL is equivalent to MML.

C. NP-hardness of MML

The decision version of MAL ($\mathbf{MAL_D}$) seeks a m-arbor in an auxiliary digraph such that its lifetime is no less than some positive bound. The problem is restated as follows.

Definition 5: Given W, the ***K*-subgraph** is derived by removing all the intra-node arcs heavier than K from the auxiliary digraph of W, where K is a positive value.

M-arbor in *K*-subgraph (AIK): Given W, $\texttt{mult}(s,D)$ and $K\in\mathbb{R}^+$, is there a m-arbor in the K-subgraph?

Since *every* possible m-arbor of lifetime $1/K$ or higher is included in the K-subgraph and the lifetime of *any* m-arbor in the K-subgraph is at least $1/K$, we have:

Observation 1: AIK and MAL_D are equivalent.

AIK's computational hardness arises from inherent multi-dimensional *selection* problem - selection of an intra-node arc in each bipartite digraph.

Lemma 2: AIK is NP-complete.

Proof: Clearly, AIK $\in$ NP. We now reduce X3C to AIK. For an arbitrary instance of X3C[2] (i.e., a set X, $|X|=3n$, and a collection C, $|C|=m$, of 3-element subsets of X), we define a WANET W and construct a K-subgraph G and prove that G has a m-arbor iff C has an exact cover for X.

WANET W is a three tiered network: the first tier has the (source) node s; the second tier (called the forwarder nodes) has n groups (referred to as C_i, $1\leqslant i\leqslant n$) of nodes, each group consisting of m nodes c_u, $1\leqslant u\leqslant m$, referred to as node c_u of group C_i, corresponding to u^{th} element of C; and the third tier consisting of $3n$ nodes x_j, $1\leqslant j\leqslant 3n$, each representing a distinct element of X. The link structure of all the n C_i groups is identical. Each node c_u of any group C_i in tier two has i) a directed incoming link from node s indicating that it can receive from s; ii) an outgoing link to its right neighbor (with wraparound i.e. c_m's right neighbor is c_1) in its group; and iii) three outgoing links to the third tier x-nodes which correspond to the elements of u^{th} set in C. The link sets E_s, E_c, and E_x consist of all the links of type i), ii), and iii), respectively. The multicast problem is $\texttt{mult}(s,X)$. Effectively, the source is separated from each destination by a group of connected *forwarder* nodes. The energy consumption can be large if a forwarder forwards a bit received from the source to a destination directly. Hence, a solution (multicast tree) involves passing a bit from one forwarder to another, to

[2]**Exact cover by 3-sets (X3C):** Given a set X, $|X|=3n$, and a collection C, $|C|=m$, of 3-element subsets of X, does C contain an exact cover for X, i.e., a set $C'\subseteq C$ such that every element of X occurs in *exactly* one member of C'?

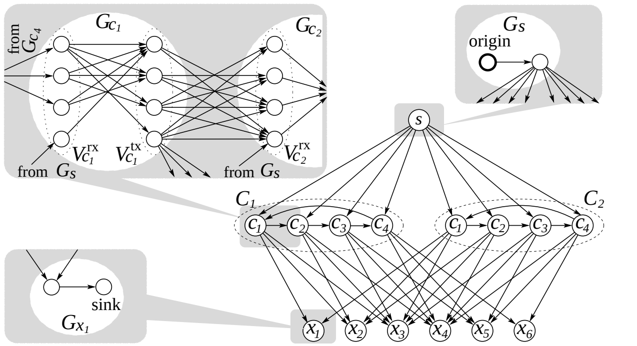


Fig. 3. Reducing X3C to AIK for n=2 and m=4. The transmission (reception) vertices in each bipartite digraph are sorted in ascending order according to the represented transmission (reception) levels.

trade the reception and the transmission energy at each hop. Specifically, we define W=$(N, E, B, \Delta, \Gamma, r, t, g)$, where:

N: $\{s\} \cup \bigcup_{i=1}^{n} C_i \cup X$ (=$\{x_j$:1⩽j⩽3$n\}$). Note $|N|$=1+3n+nm.

E: $E_s \cup E_x \cup E_c$ of directed links. Note $|E|$=5nm.

B: $B(u)$=1 for any node u∈N.

Δ, Γ: $\Delta(s)$=0, $\Gamma(s)$=1; $\Delta(c)$=$\Gamma(c)$=m, c∈C_i, 1⩽i⩽n; $\Delta(x_j)$=1, $\Gamma(x_j)$=0, 1⩽j⩽3n.

r, t: $t_s(1)$=1; $r_c(k)$=$t_c(k)$=k, c∈C_i, 1⩽i⩽n, 1⩽k⩽m; $r_{x_j}(1)$=1, 1⩽j⩽3n.

g:

$$g_{uv}(j) = \begin{cases} m, & \text{if } j = 1 \text{ and } (u,v) \in E_s \\ m - j, & \text{if } 1 \leqslant j < m \text{ and } (u,v) \in E_c \\ 1, & \text{if } j = m \text{ and } (u,v) \in E_c \\ 2 \text{ (i.e. unreachable)}, & \text{if } 1 \leqslant j < m \text{ and } (u,v) \in E_x \\ 1, & \text{if } j = m \text{ and } (u,v) \in E_x. \end{cases}$$

Graphical depiction of g_{uv} below shows it is non-increasing:

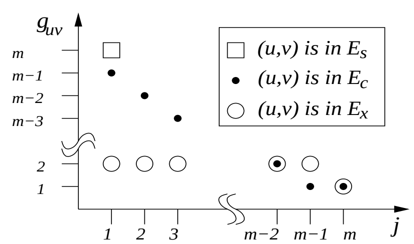


Fig. 3 depicts W for n=2 and m=4. The auxiliary digraph of W is constructed as discussed in Section III-A; the K-subgraph G is obtained by removing all the intra-node arcs **heavier than** K=m+1. Fig. 3 shows the components of G in the shadowed areas. Note G can be constructed in polynomial-time.

In G, any forwarder can only hear the source at the m^{th} level and only transmit to a destination at the m^{th} level. Due to the removal of higher energy intra-node arcs, any path connecting the origin and a sink has to span many forwarders, incrementally applying an increase in the transmission power and a decrease in the reception power. A *through-path* is a path from any m^{th} reception vertex to any m^{th} transmission vertex in C_i, 1⩽i⩽n. Forwarders have m power levels, thus a through-path traversing a forwarder group C_i has to have at least $m-1$ inter-node hops (see example in Fig. 4).

Observation 2: Since a through-path can be of $m-1$ hops, and there are m forwarders in group C_i, there exists a *flat through-path* for any sink x_i through any forwarder group C_i (for example, path $[s, c_2, c_3, c_4, c_1, x_2]$ in Fig. 3).

Observation 3: There is at most one flat through-path in each group as the above path spanning m bipartite digraphs is

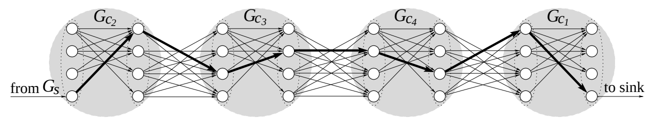


Fig. 4. An example flat through-path (bold arrows) for m=4.

the shortest through-path in terms of number of arcs.

X3C⇒AIK: w.l.o.g., let C'=$\{c_1, c_2, \ldots, c_n\}$⊆$C$ be an exact cover for X. Due to Observation 2, the arborescence in G that is rooted at the origin, spans the flat through-path incident to the m^{th} transmission vertex of G_{c_i} in C_i, 1⩽i⩽n, and covers all the sinks is a m-arbor, and thus is a solution to AIK.

X3C⇐AIK: Let H be a m-arbor in G, covering the 3n sinks. Thanks to Observation 3, H has at most one flat through-path in **each** C_i, 1⩽i⩽n. A simple counting argument on the number of end vertices covered by these flat through-paths establishes that a solution to $\texttt{mult}(s, X)$ gives an exact cover for X. ■

Theorem 1: MML is NP-hard.

Proof: AIK is NP-complete (Lemma 2), but AIK is equivalent to the decision version of MAL (Observation 1), which makes MAL NP-hard. The MML problem, being equivalent to MAL (Corollary 1), is thus NP-hard itself. ■

The NP-completeness of AIK holds when nodes have at least two transmission and two reception levels. In the case of single transmission or reception level, AIK is solvable in poly time, e.g., using Breadth-First Search, and so is MML.

IV. Additional Results

Result 1: Maximizing *broadcast* lifetime (MBL) is a special case of MML, where all non-source nodes are destinations. Notice in the proof of Lemma 2 any multicast tree for $\texttt{mult}(s, X)$ spans every node in W and is hence a broadcast tree. Hence, similarly, MBL can be proved to be NP-hard.

Result 2: Another relevant problem, the *MRBE*, as defined below, can be proved to be NP-hard using the technique in Section III: in the auxiliary digraph, the weight of each intra-node arc, e.g., $(u_i^{\text{rx}}, u_j^{\text{tx}})$, is replaced with the inverse of the amount of residual battery energy, i.e., $1/(B_u - \tau(r_u(i) + t_u(j)))$.

Maximum residual battery energy (MRBE): Given W, $\texttt{mult}(s, D)$ and the number τ of bits to be delivered to the destinations, find a multicast tree such that, after finishing the task, the minimum residual battery energy is maximized.

Result 3: Maintaining a dynamic multicast tree structure of maximal lifetime, which changes over fixed or arbitrary periods of time, is at least as hard as MML, which assumes a fixed multicast tree, and hence is NP-hard as well.

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