Understanding the solution

For a given problem instance, there may be several stable matchings.
- Do all executions of Gale-Shapley yield the same stable matching?
- If so, which one?

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an instance with two stable matchings: $S = \{ A-X, B-Y, C-Z \}$ and $S' = \{ A-Y, B-X, C-Z \}$
Understanding the solution

Def. Student $s$ is a valid partner for hospital $h$ if there exists any stable matching in which $h$ and $s$ are matched.

Ex.
- Both Xavier and Yolanda are valid partners for Atlanta.
- Both Xavier and Yolanda are valid partners for Boston.
- Zeus is the only valid partner for Chicago.

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Hospital-optimal assignment. Each hospital receives best valid partner.
  - Is it perfect?
  - Is it stable?

Claim. All executions of Gale–Shapley yield hospital-optimal assignment.
Corollary. Hospital-optimal assignment is a stable matching!
Hospital optimality

Claim. Gale–Shapley matching $S^*$ is hospital-optimal.

Pf. [by contradiction]
- Suppose a hospital is matched with student other than best valid partner.
- Hospitals propose in decreasing order of preference
- $\Rightarrow$ some hospital is rejected by valid partner during Gale–Shapley.
- Let $h$ be first such hospital, and let $s$ be the first valid student that rejects $h$.
- Let $M$ be a stable matching where $h$ and $s$ are matched.
- When $s$ rejects $h$ in Gale–Shapley, $s$ forms (or re-affirms) commitment to a hospital, say $h'$.
- $\Rightarrow$ $s$ prefers $h'$ to $h$.
- Let $s'$ be partner of $h'$ in $M$.
- $h'$ had not been rejected by any valid partner (including $s'$) at the point when $h$ is rejected by $s$.
- Thus, $h'$ had not yet proposed to $s'$ when $h'$ proposed to $s$.
- $\Rightarrow$ $h'$ prefers $s$ to $s'$.
- Thus $h' - s$ is unstable in $M$, a contradiction.  

because this is the first rejection by a valid partner
Student pessimality

Q. Does hospital-optimality come at the expense of the students?
A. Yes.

Student-pessimal assignment. Each student receives worst valid partner.


Pf. [by contradiction]
• Suppose $h - s$ matched in $M^*$ but $h$ is not the worst valid partner for $s$.
• There exists stable matching $M$ in which $s$ is paired with a hospital, say $h'$, whom $s$ prefers less than $h$.
  \[ \Rightarrow \text{s prefers } h \text{ to } h'. \]
• Let $s'$ be the partner of $h$ in $M$. By hospital-optimality, $s$ is the best valid partner for $h$.
  \[ \Rightarrow h \text{ prefers } s \text{ to } s'. \]
• Thus, $h - s$ is an unstable pair in $M$, a contradiction. ■
Deceit: Machiavelli meets Gale-Shapley

Q. Can there be an incentive to misrepresent your preference list?
   • Assume you know hospital's propose-and-reject algorithm will be run.
   • Assume preference lists of all other participants are known.

Fact. No, for any hospital; yes, for some students.
Extensions

Extension 1. Some participants declare others as unacceptable.
Extension 2. Some hospitals have more than one position.
Extension 3. Unequal number of positions and students.

Def. Matching $M$ is **unstable** if there is a hospital $h$ and student $s$ such that:
- $h$ and $s$ are acceptable to each other; and
- Either $s$ is unmatched, or $s$ prefers $h$ to assigned hospital; and
- Either $h$ does not have all its places filled, or $h$ prefers $s$ to at least one of its assigned students.
Historical context

National resident matching program (NRMP).
- Centralized clearinghouse to match med-school students to hospitals.
-Began in 1952 to fix unraveling of offer dates.
-Originally used the “Boston Pool” algorithm.
-Algorithm overhauled in 1998.
  - med-school student optimal
  - deals with various side constraints
    (e.g., allow couples to match together)

The Redesign of the Matching Market for American Physicians:
Some Engineering Aspects of Economic Design

By Alvin E. Roth and Elliott Peranson*

We report on the design of the new clearinghouse adopted by the National Resident Matching Program, which annually fills approximately 20,000 jobs for new physicians. Because the market has complementarities between applicants and between positions, the theory of simple matching markets does not apply directly. However, computational experiments show the theory provides good approximations. Furthermore, the set of stable matchings, and the opportunities for strategic manipulation, are surprisingly small. A new kind of “core convergence” result explains this; that each applicant interviews only a small fraction of available positions is important. We also describe engineering aspects of the design process. (JEL C78, B41, J44)
2012 Nobel Prize in Economics


COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE
D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

1. Introduction. The problem with which we shall be concerned relates to the following typical situation: A college is considering a set of $n$ applicants of which it can admit a quota of only $q$. Having evaluated their qualifications, the admissions office must decide which ones to admit. The procedure of offering admission only to the $q$ best-qualified applicants will not generally be satisfactory, for it cannot be assumed that all who are offered admission will accept.

Alvin Roth. Applied Gale–Shapley to matching med-school students with hospitals, students with schools, and organ donors with patients.
A modern application

Content delivery networks. Distribute much of world's content on web.

User. Prefers web server that provides fast response time.
Web server. Prefers to serve users with low cost.
Goal. Assign billions of users to servers, every 10 seconds.

Algorithmic Nuggets in Content Delivery

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This article is an editorial note submitted to CCR. It has NOT been peer reviewed.
The authors take full responsibility for this article's technical content. Comments can be posted through CCR Online.

ABSTRACT
This paper "peeks under the covers" at the subsystems that provide the basic functionality of a leading content delivery network. Based on our experiences in building one of the largest distributed systems in the world, we illustrate how sophisticated algorithmic research has been adapted to balance the load between and within server clusters, manage the caches on servers, select paths through an overlay routing network, and elect leaders in various contexts. In each instance, we first explain the theory underlying the algorithms, then introduce practical considerations not captured by the theoretical models, and finally describe what is implemented in practice. Through these examples, we highlight the role of algorithmic research in the design of complex distributed systems. The paper also illustrates the close synergy that exists between research and industry where research ideas cross over into products and product requirements drive future research.